Solutions: Homework 2

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Problem 1. Let (X, d) be a complete metric space and $Y \subset X$ is closed. Show that (Y, d) is a complete metric space.

Proof. Let $\{F_n\}$ be a sequence of non-empty closed sets in Y with diam $F_n \to 0$. Since Y is closed in X, the F_n's are also closed in X. Then, by Cantor's theorem, the completeness of X implies that $\cap_{n=1}^{\infty} F_n$ consists of a single point. Since this holds for any such sequence in Y, Cantor's theorem implies that (Y, d) is complete. $\square$

Problem 2. Show that diam $A =$ diam $\overline{A}$.

Proof. Since $\{d(x, y) : x \text{ and } y \text{ in } A\} \subset \{d(x, y) : x \text{ and } y \text{ in } \overline{A}\}$, we have diam $A \leq$ diam $\overline{A}$. We will now show that for all $\epsilon > 0$, there exists $x, y \in A$ such that $d(x, y) >$ diam $\overline{A} - \epsilon$. This would show that diam $A \geq$ diam $\overline{A}$, which completes the proof. Choose $x_0, y_0 \in \overline{A}$ such that $d(x_0, y_0) >$ diam $\overline{A} - \epsilon/2$. Since $x_0, y_0 \in \overline{A}$, we can find $x, y \in A$ such that $d(x, x_0) < \epsilon/4$ and $d(y, y_0) < \epsilon/4$. Then, using the triangle inequality, we have $d(x, x_0) + d(x, y) + d(y, y_0) \geq d(x_0, y_0) >$ diam $\overline{A} - \epsilon/2$. So $d(x, y) >$ diam $\overline{A} - \epsilon/2 - d(x, x_0) - d(y, y_0) >$ diam $\overline{A} - \epsilon/2 - \epsilon/4 - \epsilon/4 =$ diam $\overline{A} - \epsilon$. $\square$

Problem 3. Suppose $\{x_n\}$ is a Cauchy sequence and $\{x_{n_k}\}$ is a subsequence which is convergent. Show that $\{x_n\}$ must be convergent.

Proof. Let $x_{n_k} \to x$ in (X, d). Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon/2 \; \forall \; n, m \geq N$. Choose $n_k \geq N$ such that $d(x_{n_k}, x) < \epsilon/2$. Then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon/2 + \epsilon/2 = \epsilon$. So $d(x_n, x) < \epsilon \; \forall \; n \geq N$. This implies that $x_n \to x$ in X. $\square$

Problem 4. Let $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ be points in $\mathbb{R}^n$ with $p_k < q_k$ for each k. Let $R = [p_1, q_1] \times ... \times [p_n, q_n]$ and show that

$$\text{diam } R = d(p, q) = [\sum_{k=1}^{n} (q_k - p_k)^2]^{\frac{1}{2}}.$$

Proof. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be two points in R. Then $d(x, y) = [\sum_{k=1}^{n} (x_k - y_k)^2]^{\frac{1}{2}}$. Since $x_k, y_k \in [p_k, q_k] \; \forall \; k, |x_k - y_k| \leq q_k - p_k$, hence $(x_k - y_k)^2 \leq (q_k - p_k)^2 \; \forall \; k$. Adding over all k, we get $d(x, y) \leq d(p, q) \; \forall \; x, y \in R$. So diam $R \leq d(p, q)$. Since $p, q \in R$, diam $R \geq d(p, q)$. So diam $R = d(p, q)$. $\square$

Problem 5. Let $F = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$ and let $\epsilon > 0$; use Problem 4 to show that there are rectangles $R_1, ..., R_m$ such that $F = \cup_{k=1}^m R_k$ and diam $R_k < \epsilon$ for each k. If $x_k \in R_k$ then it follows that $R_k \subset B(x_k; \epsilon)$.

Proof. For all $k = 1, ..., n$, choose a positive number $r_k < \epsilon/\sqrt{n}$ such that $\frac{b_k - a_k}{r_k}$ is a natural number. Let $S_k = \{[a_k + jr_k, a_k + (j+1)r_k] : j = 0, ..., \frac{b_k - a_k}{r_k} - 1\}$ for $k = 1, ..., n$. Let $R_1, ..., R_m$ be all the rectangles of the form $A_1 \times ... \times A_n$ with $A_k \in S_k \ \forall \ k = 1, ..., n$. Then, by Problem 4, diam $R_j = (\sum_{k=1}^n r_k^2)^{\frac{1}{2}} < (\sum_{k=1}^n (\epsilon/\sqrt{n})^2)^{\frac{1}{2}} = \epsilon$. Also, clearly $F = \cup_{k=1}^m R_k$. If $x_k \in R_k$ and $x \in R_k$, then $d(x_k, x) \leq$ diam $R_k < \epsilon \implies R_k \subset B(x_k; \epsilon)$. $\square$